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GAMES IN EXTENDED AND NORMAL FORM AND MATHEMATICAL PROGRAMMING

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ABSTRACT

Games in normal form and mixed extensions of these games are considered in this paper. The transformation of games in extensive form into games in normal form is treated informally by means of examples. A survey of minimax theorems and equilibrium point theorems is given. Attention is paid to the relation between matrix game theory and linear programming theory and also between bimatrix game theory and linear complementarity problems.

1. INTRODUCTION

Game theoretists are engaged in the design and the study of mathematical models for conflict situations, i.e. situations in which some decision makers (players) with diverse objectives are involved. The subject owes its name to one of its inspiration sources, namely the class of parlour games (poker, nim, chess,...). Furthermore, there is a fruitful interaction between game theory and the following subjects: mathematical programming, dynamic programming, control theory, mathematical economics, statistics, sociology and military science.

The foundation of Game Theory was laid by JOHN VON NEUMANN (1903-1957) in his beautiful paper *Zur Theorie der Gesellschaftsspiele,* Math. Ann. 100, 1928, 295-320. The subject obtained reputation in a wider circle, after the appearance in 1944 of the book *Theory of Games and Economic Behavior* (Princeton University Press), which John von Neumann wrote together with the mathematical economist OSKAR MORGENSTERN (1902-1977). Since that time some thousands of papers connected with the subject appeared in various places. The need of game theorists to have their own journal was fulfilled in 1971 by the foundation of *International Journal of Game Theory* (Physica Verlag, Vienna).

It is not to be wondered that not all conflict situations can be caught in one model. It is natural to make a subdivision in classes of games by looking at the various aspects, which are relevant in conflict situations such as

(a) the number of decision makers (two-person games, games with more than two players),
(b) the admissibility of forming coalitions, yes or no (cooperative games, non-cooperative games),
(c) the number of moves (games in normal form, games in extended form, stochastic games, differential games),
(d) the number of strategies available (finite games, infinite games),
(e) the source of the payoffs (zero-sum games, non-zero-sum games),
(f) the amount of information which a player has available when it is his move (games with and without perfect information).

For a very detailed attempt to classify games we refer to VOROB'EV [36]. The most widely studied classes of games are

- games in normal form
- games in extended form
- games in characteristic function form (cooperative games)
- stochastic games
- differential games.

In this survey paper we occupy ourselves mainly with the class of non-cooperative two-person games in normal form, where the subclass of zero-sum games will obtain special attention. After having introduced a number of basic notions, we will concentrate on problems such as

(a) the existence of equilibrium points (or ε-equilibrium points for each $\varepsilon > 0$) of non-zero-sum games; the existence of value and optimal strategies for zero-sum games.
(b) the existence of (ε)-equilibrium points for mixed extensions of games in normal form.
(c) the calculation of value and optimal strategies (or equilibrium points, respectively) for zero-sum games (non-zero-sum games, respectively). Herewith linear programming problems and linear

complementary problems come up for discussion.

(d) the structure of optimal strategy spaces (sets of equilibrium points).

In Section 3 we indicate in an informal manner how games in extended form can be transformed into games in normal form. For works, in which games in normal form obtain detailed attention we refer to BERGE [2], BLACKWELL & GIRSHICK [3], BURGER [5], KARLIN [12,13], LUCE & RAIFFA [17], VON NEUMANN & MORGENSTERN [21], OWEN [23], PARTHASARATHY & RAGHAVAN [24], TIJS [31], VOROB'EV [37] and WALD [39].

2. GAMES IN NORMAL FORM

A *two-person game (in normal form)* is an ordered quadruplet $\langle X,Y,K_1,K_2\rangle$, in which X and Y are non-empty sets and where $K_1: X \times Y \to \mathbb{R}$ and $K_2: X \times Y \to \mathbb{R}$ are realvalued functions on $X \times Y$. X, Y, K_1, K_2 are called respectively: *(pure) strategy space of player* I, *(pure) strategy space of player* II, *payoff function of player* I, *payoff function of player* II. The elements of X (Y) are called *(pure) strategies of player* I (II).

A play of such a game runs as follows: two players, called I and II, choose independently of each other an element x and y from the sets X and Y, respectively, after which follows a payoff of $K_1(x,y)$ units to player I and a payoff of $K_2(x,y)$ units to player II.

A two-person game $\langle X,Y,K_1,K_2\rangle$ is called a *zero-sum game*, if $K_1 + K_2 = 0$. In that case we denote the game also by the ordered triplet $\langle X,Y,K\rangle$, where $K := K_1 = -K_2$. In a zero-sum game the one player pays the other.

A two-person game $\langle X,Y,K_1,K_2\rangle$ is called a *finite game* if both sets X and Y are finite sets, and a *semi-infinite (infinite) game* if one of the strategy spaces contains (both strategy spaces contain) an infinite number of elements. If $X = Y = [0,1]$, then we call it a *game on the square*.

Let $A = [a_{ij}]_{i=j}^{m}{}_{,j=1}^{n}$, $B = [b_{ij}]_{i=1}^{m}{}_{,j=1}^{n}$ be two m×n-matrices $(m,n \in \mathbb{N})$ of real numbers. Then the two-person game $\langle X,Y,K_1,K_2\rangle$ with $X = \{1,2,\ldots,m\}$, $Y = \{1,2,\ldots,n\}$ and $K_1(i,j) = a_{ij}$, $K_2(i,j) = b_{ij}$ for all $(i,j) \in X \times Y$, is called the (finite) *bimatrix game with payoff* *matrices* A and B. We denote this game by (A,B). If B = -A, then we denote the game by A and call A the *payoff matrix of the matrix game* A. It is obvious that each finite two-person game $<X,Y,K_1,K_2>$ can be seen as a finite bimatrix game, by numbering the elements of X and Y. Likewise, semi-infinite (infinite) games with countable strategy spaces correspond with semi-infinite (infinite) bimatrix games.

EXAMPLE 1 (*Duopoly model of A. Wald*). Suppose a particular product is produced by two producers called I and II, which have production capacity $c_1 > 0$ and $c_2 > 0$, respectively. If I decides to produce a quantity $x \in [0,c_1]$ per time unit and to bring it on the market and II a quantity $y \in [0,c_2]$, then we suppose that the product yields a price $p(x+y) \geq 0$ per unit. Suppose further that the production costs are $k_1(x)$ (resp. $k_2(x)$) if I (II) produces an amount x (y). Then this market situation can be converted into an infinite two-person game $<X,Y,K_1,K_2>$, where $X = [0,c_1]$, $Y = [0,c_2]$, and $K_1(x,y) = xp(x+y) - k_1(x)$, $K_2(x,y) = yp(x+y) - k_2(y)$.

EXAMPLE 2 (*A game of timing*). In a duel two "players" I and II have pistols with silencers. The rules of the duel are as follows: only in the time interval [0,1] one may shoot, at most once. If one of the players is shot by the opponent, then he may not shoot any more and has to pay one unit (of money) to that opponent. We suppose equal duel capacity: the probability of hitting at time t will be denoted by p(t). This situation can be reduced to an infinite zero-sum two-person game (on the square) $<X,Y,K_1,K_2>$, where $X = Y = [0,1]$ and

$$\begin{aligned} K_1(x,y) &= p(x) - (1-p(x))p(y) && \text{if } x < y \\ &= 0 && \text{if } x = y \\ &= -p(y) + (1-p(y))p(x) && \text{if } x > y \end{aligned}$$

and

$$K_2(x,y) = -K_1(x,y) \qquad \text{for each } (x,y) \in [0,1] \times [0,1].$$

EXAMPLE 3 (*An advertising campaign model*). Two producers I and II of the same product (who may not cooperate), each control half of the market at some moment, which yields 8 units per month for each of them. Suppose that both parties have to decide independently of each other whether to start an advertising campaign or not, which costs

2 units. Call the possible decisions S and NS, respectively. Suppose that the share in the market in the following month (we do not look further to make life easy!) does not change if both parties choose S or both choose NS and that in the other two cases, that party which starts an advertising campaign obtains 75 percent of the market. This situation can be modelled as a finite two-person game $\langle X,Y,K_1,K_2\rangle$, where $X = Y = \{NS,S\}$, $K_1(NS,NS) = 8$, $K_1(NS,S) = 4$, $K_1(S,NS) = 10$, $K_1(S,S) = 6$, $K_2(NS,NS) = 8,\ldots$ and hence also as a bimatrix game (A,B), where

$$A = \begin{array}{c} \begin{array}{cc} NS & S \end{array} \\ \begin{bmatrix} 8 & 4 \\ 10 & 6 \end{bmatrix} \end{array} \begin{array}{c} \\ NS \\ S \end{array}, \qquad B = \begin{array}{c} \begin{array}{cc} NS & S \end{array} \\ \begin{bmatrix} 8 & 10 \\ 4 & 6 \end{bmatrix} \end{array} \begin{array}{c} \\ NS \\ S \end{array},$$

or shorter

$$(A,B) = \begin{bmatrix} (8,8) & (4,10) \\ (10,4) & (6,6) \end{bmatrix}.$$

<u>EXAMPLE 4</u>. The following curious game (by A. Wald) is played as follows: players I and II mention independently of each other a natural number. Then the player who has called the highest number obtains 1 unit from the other, while no payoff is made when both have called the same number. This game corresponds with the infinite matrix game $[a_{ij}]_{i=1,j=1}^{\infty\ \ \infty}$, where $a_{ij} = 1$ if $i > j$, $a_{ij} = -1$ if $i < j$ and $a_{ii} = 0$.

<u>EXAMPLE 5</u>. Let $S \subset \mathbb{N}$. Let $\langle\{1,2\},\mathbb{N},K\}$ be the zero-sum game with $K(i,j) = 1$ if $i+j \in S$ and otherwise $K(i,j) = -1$.
If e.g. $S = \{3,6,9,\ldots\}$, then this game corresponds with the

$$2\times\infty\text{-matrix game} \begin{bmatrix} -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & \cdot & \cdot & \cdot \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & \cdot & \cdot & \cdot \end{bmatrix}.$$

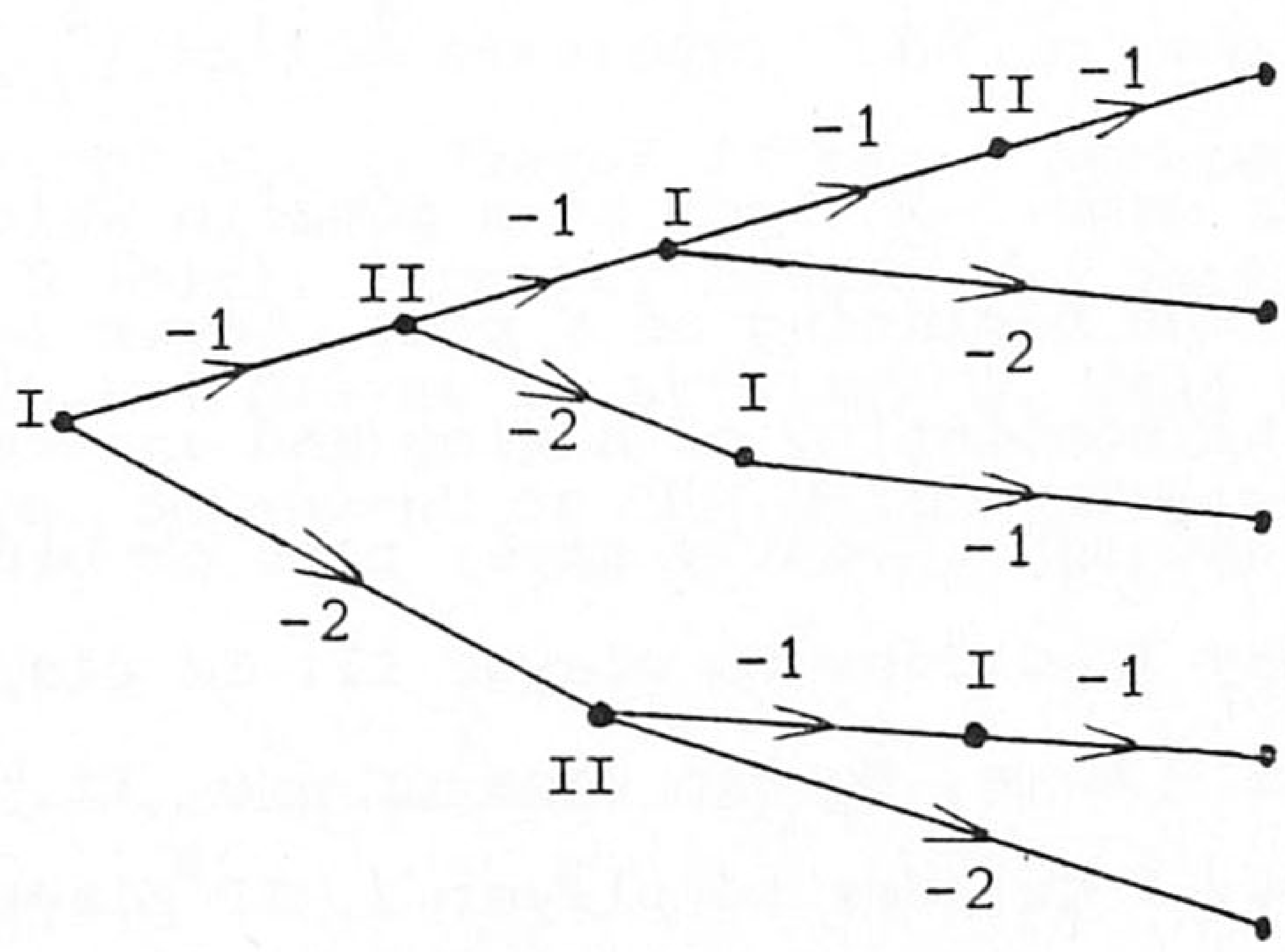
3. THE NORMALIZING OF GAMES IN EXTENDED FORM

In a two-person game in normal form both players make exactly one move independently of each other. In parlour games, usually players make more than one move and the moves are not independently of the other players. Sometimes also chance plays a role (e.g. dealing of cards). John von Neumann noticed that in principle most of the parlour games can be reduced to games in normal form (and often even to finite matrix games). The idea hereby is roughly speaking, to understand by a *strategy* of a player: a completely worked out playing plan beforehand, which tells the player (or a deputy) exactly what to do in each situation of each possible play, in which the player has to make a move. If both players each choose such a playing plan and give it to an arbiter, then the latter can calculate to which payoffs the chosen strategies correspond.

The parlour games above, can be seen as examples of *games in extended form,* such as introduced by KUHN [15].

Before one can say exactly what games in extended form are, one has to introduce notions as: directed graph, (game) tree with a root, partition of the alternatives, players partition, chance move, move of a player, information partition, play and payoff function. Subsequently, one can then define what the strategy is for a player in such a game and show that those games can be reduced to finite games in normal form. It should cost too much time and it is probably not worth while, to follow here this formal way in treating games in extended form. Therefore, we try to demonstrate the normalizing procedure by some examples.

EXAMPLE 6 (*Nim game with one pile*).



Payoff to I	Payoff to II
-1	+1
+1	-1
+1	-1
+1	-1
-1	+1

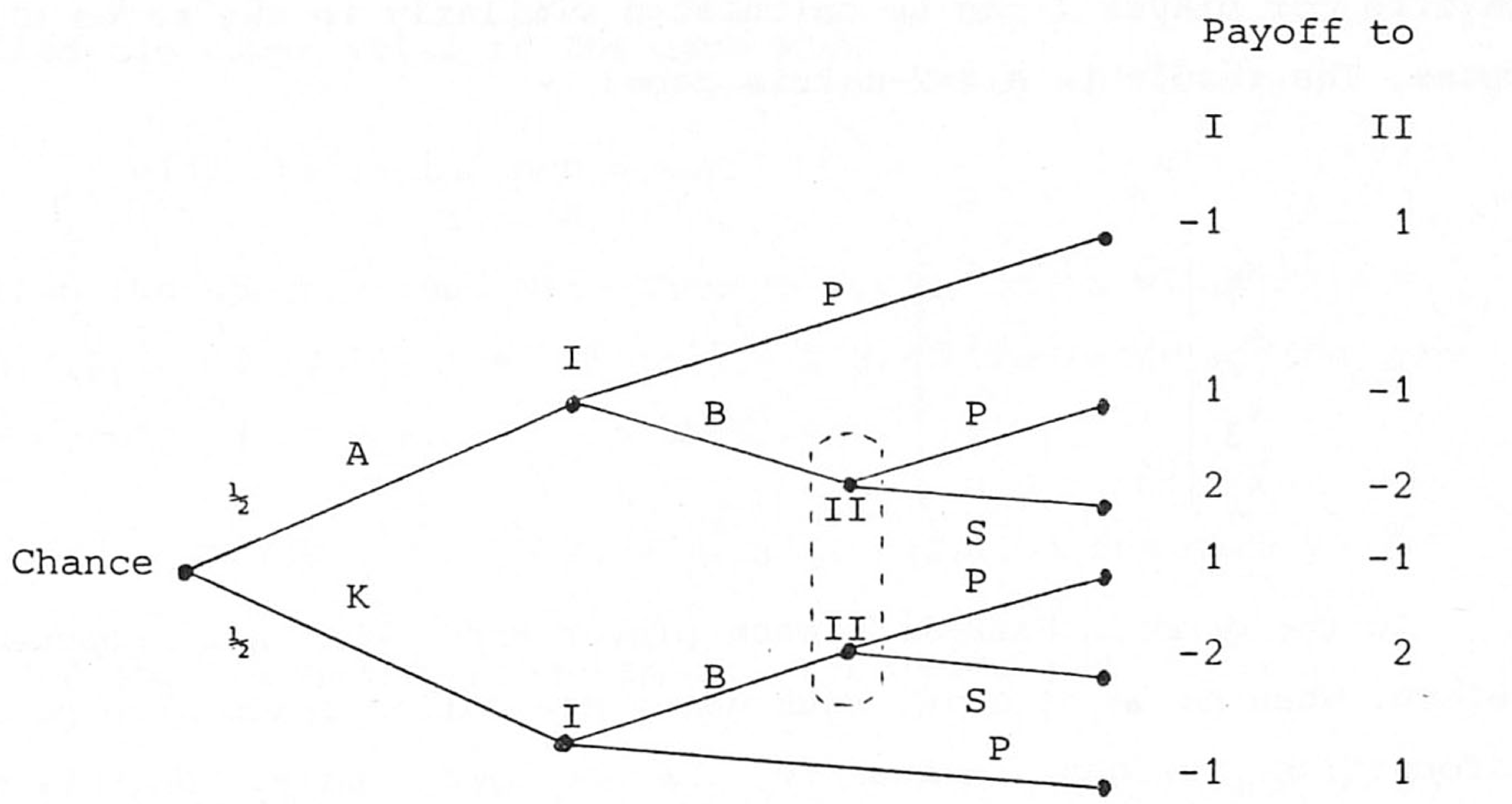
Fig. 1. Nim with one pile

There is one pile of n matches ($n \in \mathbb{N}$). The players have to take on there turn 1 or 2 matches of the pile until all matches are away. Player I starts and the player who takes the last match wins. In the case that n = 4 we can visualize the game as in Figure 1. In that figure at the vertices, corresponding with moves, the player is indicated, whose move it is. At the vertices corresponding with the end of a play the payoffs to the players are notated. A number -1 (-2) corresponds with taking 1 match (2 matches). We are going to normalize this game. Therefore, we note first that player I has available three strategies, which we will denote by (1;1), (1;2) and (2). Hereby we mean by (2) the strategy: "take in the first move 2 matches and take the last match if one has to make still a move". With (1;j), where j = 1,2, we mean the following playing plan: "take at the first move 1 match; then if player II takes 1 match, then take in the second move j matches; if player II takes after the first move 2 matches, then take the last match". Player II has available 4 strategies, which we will denote by (i,j), where $i,j \in \{1,2\}$, and in which (i,j) is the following playing plan for player II: "take i matches, if player I took in his first move 1 match, and take j matches if player II took 2 matches (and finish if necessary the game in a trivial manner)". It will be clear that the above game corresponds with the 3×4-matrix game with the following payoff matrix:

$$\begin{array}{c c} & \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{array}{c} (1;1) \\ (1;2) \\ (2) \end{array} & \left[\begin{array}{cccc} -1 & -1 & +1 & +1 \\ +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \end{array}\right] \end{array}$$

<u>EXAMPLE 7</u> (*A simple poker game*). We look at a game in which a chance element plays a role. At the beginning of a play player I gets one of the cards in a stock {K,A} consisting of a king and an ace. Player I looks at the given card and subsequently says: pass or bid. If player I passes, then he has to pay 1 guilder to player II. If player I bids, then it becomes player II's move. He can pass or see. If player II passes, then he has to pay 1 guilder to player I. If player II sees, then he gets 2 guilders from player I, if player I has a king and he pays 2 guilders to player I, if player I has an ace. For a schematic

diagram look at Figure 2. In this two-person game player II is not completely informed about the course of the play when it is his turn.



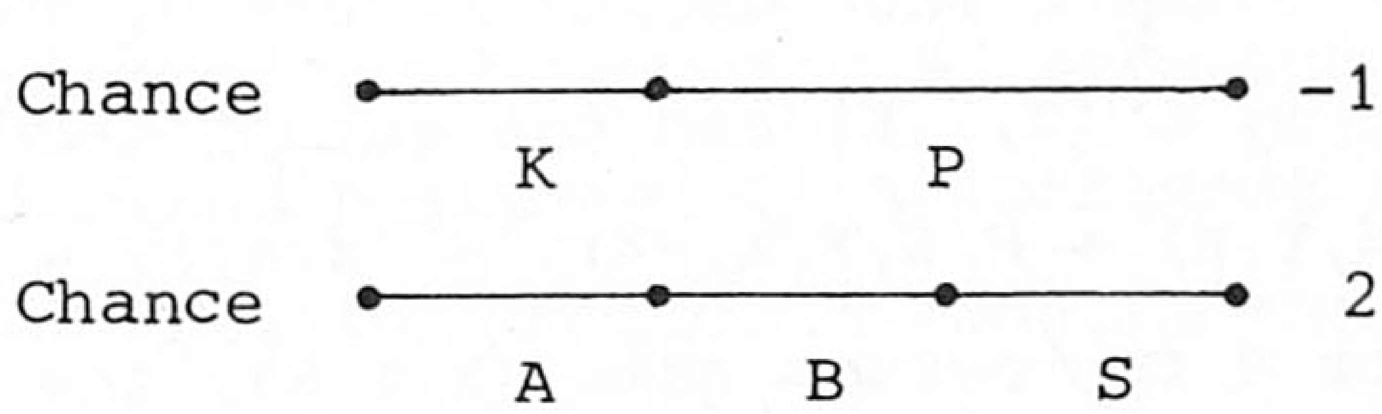
(P = pass, B = bid, S = see)

Fig.2. A simple poker game

He knows what player I did but not what chance (also called the chance player) did at the beginning of the play. (In Figure 2 this is indicated by dots.) We now are going to put this game into the normal form. We indicate the four strategies of player I by

$$x_1 = \binom{A\ K}{P\ P},\ x_2 = \binom{A\ K}{P\ B},\ x_3 = \binom{A\ K}{B\ P},\ x_4 = \binom{A\ K}{B\ B},$$

where e.g. $\binom{A\ K}{B\ P}$ is the strategy: "bid in case an ace is obtained and pass with a king" etc.. Player II has 2 strategies, which we denote by P (pass) and S (see). Suppose, before the match player I decides to use strategy $\binom{A\ K}{B\ P}$ and player II strategy S. Then this can result in two possible plays, dependent of the chance mechanism, both with probability $\frac{1}{2}$:



After the first play a payoff -1 follows, after the second play a payoff 2 from player II to player I. The expected payoff for player I corresponding with the above strategies is: $\frac{1}{2}(-1)+\frac{1}{2}(2) = \frac{1}{2}$. The expected payoffs for player I can be calculated similarly in the seven other cases. The result is a 4×2-matrix game:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} P & \quad S \end{array} \\ \begin{bmatrix} -1 & -1 \\ 0 & -1\frac{1}{2} \\ 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \end{array}$$

In the game in Example 6 each player knows what has happened before, when he is at move. Such games are called *games with perfect information*. The game in Example 7 is not such a game. For this game the question: "How to play wisely?" cannot be answered immediately, while that can be done for the game with perfect information in Example 6; in that game strategy (1;2) is a winning strategy for player I.

4. EQUILIBRIUM POINTS OF GAMES IN NORMAL FORM

We start with some definitions. Let $\langle X,Y,K_1,K_2\rangle$ be a two-person game and $\varepsilon \geq 0$. A point $(x^*,y^*) \in X\times Y$, such that

$$K_1(x^*,y^*) \geq \sup_{x\in X} K_1(x,y^*)-\varepsilon, \quad K_2(x^*,y^*) \geq \sup_{y\in Y} K_2(x^*,y)-\varepsilon$$

is called an *ε-equilibrium point* if $\varepsilon > 0$ and an *equilibrium point* if $\varepsilon = 0$. The set of ε-equilibrium points will be denoted by $E^\varepsilon(X,Y,K_1,K_2)$ and the set of equilibrium points by $E(X,Y,K_1,K_2)$.

For a two-person zero-sum game $\langle X,Y,K\rangle$ and $\varepsilon \geq 0$ a point $(x^*,y^*) \in X\times Y$ with

$$K(x,y^*) - \varepsilon \leq K(x^*,y^*) \leq K(x^*,y) + \varepsilon \quad \text{for all } (x,y) \in X\times Y$$

is called an *ε-saddlepoint* if $\varepsilon > 0$ and a *saddlepoint* if $\varepsilon = 0$. The set of ε-saddlepoints will be denoted by $S^\varepsilon(X,Y,K)$ and the set of saddlepoints by $S(X,Y,K)$. Note that $S(X,Y,K) = E(X,Y,K,-K)$, $S^\varepsilon(X,Y,K) = E^\varepsilon(X,Y,K,-K)$ for each $\varepsilon > 0$. For a two-person game $\langle X,Y,K\rangle$, the expression

$$\underline{v}(X,Y,K) := \sup_{x\in X} \inf_{y\in Y} K(x,y)$$

is called the *lower value of the game* and

$$\bar{v}(X,Y,K) := \inf_{y\in Y} \sup_{x\in X} K(x,y)$$

is called the *upper value*. Note that $-\infty \le \underline{v}(X,Y,K) \le \bar{v}(X,Y,K) \le \infty$. If $\underline{v}(X,Y,K) = \bar{v}(X,Y,K)$, then we call $\underline{v}(X,Y,K)$ the *value of the game* and we denote it by $v(X,Y,K)$. In that case

$$O_I(X,Y,K) := \{x^* \in X;\ K(x^*,y) \ge v(X,Y,K) \text{ for each } y \in Y\}$$

is called the *optimal strategy space of player* I and

$$O_{II}(X,Y,K) := \{y^* \in Y;\ K(x,y^*) \le v(X,Y,K) \text{ for each } x \in X\}$$

the *optimal strategy space of player* II.

REMARKS.

(a) For the 2×2-bimatrix game in Example 3 the point $(S,S) \in \{NS,S\}\times\{NS,S\}$ is the unique equilibrium point.

(b) For the $\infty\times\infty$-matrix game in Example 4, the lower value equals -1 and the upper value equals $+1$.

(c) The $1\times\infty$-matrix game $[1\ \ \frac{1}{2}\ \ \frac{1}{3}\ \ \ldots]$ has value 0. The optimal strategy space of player II is empty. There exist ε-saddlepoints for each $\varepsilon > 0$.

(d) The matrix game in Example 6 has value 1. The optimal strategy spaces of the players I and II are $\{2\}$ and $\{1,2,3,4\}$, respectively.

(e) The finite matrix game in Example 7 has no value.

In view of (d) and (e) above, we note that KUHN [15, p.209] proved that games in normal form, which are derived from games in extended form with perfect information, have at least one equilibrium point. For extensions of this result we refer to TIJS [31, p.26].

A beautiful property of zero-sum games is the fact that the equilibrium point set is a Cartesian product of two sets and that different equilibrium points lead to the same payoffs. Precisely stated: Let $\langle X,Y,K_1,K_2\rangle$ be a zero-sum game. If (x_1,y_1) and (x_2,y_2) are equilibrium points of the game, then

$$K_1(x_1,y_1) = K_1(x_2,y_2), \qquad K_2(x_1,y_1) = K_2(x_2,y_2),$$

$$(x_1,y_2),(x_2,y_1) \in E(X,Y,K_1,K_2).$$

Non-zero-sum games may fail to have such properties, as the following 2×2-bimatrix game shows: $\begin{bmatrix} (1,2) & (0,0) \\ (0,0) & (2,1) \end{bmatrix}$.

We collect some simple facts in the following three propositions.

PROPOSITION 1. *For a two-person zero-sum game* $\langle X,Y,K\rangle$ *the following two assertions are equivalent:*

(i) $\langle X,Y,K\rangle$ *has a finite value* ($v(X,Y,K) \in \mathbb{R}$);

(ii) $S^{\varepsilon}(X,Y,K) = E^{\varepsilon}(X,Y,K,-K) \neq \emptyset$ *for each* $\varepsilon > 0$.

PROOF. See TIJS [32, p.756].

PROPOSITION 2. *For a two-person zero-sum game* $\langle X,Y,K\rangle$ *the following three assertions are also equivalent (cf.* [35]*):*

(i) $\max_{x\in X}\inf_{y\in Y} K(x,y)$ *and* $\min_{y\in Y}\sup_{x\in X} K(x,y)$ *exist and are equal;*

(ii) *there exist* $v \in \mathbb{R}$, $x^* \in X$ *and* $y^* \in Y$, *such that* $K(x^*,y) \geq v$ *for each* $y \in Y$ *and* $K(x,y^*) \leq v$ *for each* $x \in X$;

(iii) $S(X,Y,K) \neq \emptyset$.

PROPOSITION 3. *Let* $\langle X,Y,K\rangle$ *be a two-person zero-sum game with value* v *and let* $\varepsilon > 0$. *Then* $S(X,Y,K) = O_I(X,Y,K) \times O_{II}(X,Y,K)$. *Furthermore,*

$$O_I^{\varepsilon}(X,Y,K) \times O_{II}^{\varepsilon}(X,Y,K) \subset S^{\varepsilon}(X,Y,K) \subset O_I^{2\varepsilon}(X,Y,K) \times O_{II}^{2\varepsilon}(X,Y,K),$$

where

$$O_I^{\varepsilon}(X,Y,K) := \{x^* \in X;\ K(x^*,y) \geq v - \varepsilon \text{ for each } y \in Y\},$$

$$O_{II}^{\varepsilon}(X,Y,K) := \{y^* \in Y;\ K(x,y^*) \leq v + \varepsilon \text{ for each } x \in X\}.$$

In so called *minimax theorems*, conditions are put on strategy spaces and payoff functions, which are sufficient to guarantee the existence of the value of zero-sum games. The first minimax theorem of John von Neumann is dated in 1928. Since that time many other minimax theorems are derived, among others by VILLE [34], FAN [8], SION [27], KÖNIG [14], TERKELSEN [30],...

To give an impression of the nature of the conditions in such theorems, we formulate here, without explication of terms and without proof, two of these theorems.

THEOREM 1 (*Minimax theorem of M. SION* [27]). *Let* $\langle X,Y,K\rangle$ *be a two-person zero-sum game, in which* X *and* Y *are convex subsets of topological vector spaces and in which* Y *is compact. Let* $K: X\times Y \to \mathbb{R}$ *be semicontinuous and quasi-concave-convex. Then the game has a value.*

THEOREM 2 (*Minimax theorem of K. FAN* [8]). *Let* $\langle X,Y,K\rangle$ *be a two-person zero-sum game, in which* X *and* Y *are compact sets and let* $K: X\times Y \to \mathbb{R}$ *be a semicontinuous concave-convex-like function. Then the game has a value and both players have optimal strategies.*

In so called (ε-)*equilibrium point theorems* one is concerned with the existence of (ε-)equilibrium points (for each $\varepsilon > 0$) of arbitrary two-person games in normal form. It follows from Proposition 1 that deriving ε-equilibrium point theorems is a natural extension of deriving minimax theorems. We formulate here only the well known equilibrium point theorem of NIKAIDÔ-ISODA [22]. For more information about ε-equilibrium point theorems we refer to RUPP [25] and TIJS [31].

THEOREM 3 (*Equilibrium point theorem of H.NIKAIDÔ & K. ISODA* [22]). *Let* $\langle X,Y,K_1,K_2\rangle$ *be a two-person game in which* X *and* Y *are compact convex subsets of topological vector spaces and in which* K_1 *and* K_2 *are continuous functions such that*

$$x \longmapsto K_1(x,y) \text{ is a concave function on } X \text{ for each } y \in Y, \text{ and}$$
$$y \longmapsto K_2(x,y) \text{ is a concave function on } Y \text{ for each } x \in X.$$

Then $E(X,Y,K_1,K_2) \neq \emptyset$.

5. MIXED EXTENSIONS

As we have seen from some of the examples in Section 4 not for all zero-sum games the lower value equals the upper value. This fact was the direct motive for J. von Neumann to introduce mixed extensions for finite zero-sum games, in the hope to bridge the gap between lower and upper value. In this section we introduce and study mixed extensions for finite and infinite zero-sum and non-zero-sum games.

Let $A = [a_{ij}]_{i=1,j=1}^{m\ \ n}$, $B = [b_{ij}]_{i=1,j=1}^{m\ \ n}$ be two m×n-matrices of real numbers ($m,n \in \mathbb{N}$). Then the two-person game $\langle S^m, S^n, E_A, E_B\rangle$ in which

$$S^m := \{p = (p_1,\ldots,p_m) \in \mathbb{R}^m;\ p \geq 0,\ \sum_{i=1}^{m} p_i = 1\}$$

$$S^n := \{q = (q_1,\ldots,q_n) \in \mathbb{R}^n;\ q \geq 0,\ \sum_{j=1}^{n} q_j = 1\},$$

$$E_A(p,q) := \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j = pAq^t,$$

$$E_B(p,q) := \sum_{i=1}^{m}\sum_{j=1}^{n} p_i b_{ij} q_j = pBq^t \qquad (p \in S^m,\ \ q \in S^n),$$

is called the *mixed extension of the (finite) bimatrix game* (A,B). The elements of S^m (S^n) are called the *mixed strategies of player* I (II). The set of equilibrium points of $\langle S^m,S^n,E_A,E_B\rangle$ is denoted by E(A,B) in the following. The two-person zero-sum game $\langle S^m,S^n,E_A\rangle$, in which E_A is defined as above, is called the *mixed extension of the (finite) matrix game* A. Value and spaces of optimal strategies of this game $\langle S^m,S^n,E_A\rangle$ will be denoted by v(A), $O_I(A)$ and $O_{II}(A)$.

For an m×n-bimatrix (A,B) a mixed strategy $(p_1,\ldots,p_m) \in S^m$ can be realized by using a chance mechanism, which chooses an element of the pure strategy space $\{1,2,\ldots,m\}$ of player I, in such a way that (row) i is chosen with probability p_i. If, in a play, player I uses the mixed strategy p and player II the mixed strategy q, then $E_A(p,q)$ and $E_B(p,q)$ can be interpretated as the expected payoffs to players I and II. It is obvious that we can identify the pure strategy $i \in \{1,\ldots,m\}$ of player I with the mixed strategy $e_i \in S^m$, where e_i is the i-th standard basis vector in $\mathbb{R}^m$, and this explains the term "extension". A fundamental result in game theory is the following

THEOREM 4 (*Minimax theorem of J. VON NEUMANN* [20]). *Let* A *be an* m×n-*matrix game. Then the mixed extension of* A *has a value and* $O_I(A) \neq \emptyset$, $O_{II}(A) \neq \emptyset$.

This theorem can e.g. be derived from the following theorem. (For another proof, see Section 6.)

THEOREM 5 (*Equilibrium point theorem of J.F. NASH* [19]). *For each* $m\times n$-*bimatrix game* (A,B) *we have:* $E(A,B) \neq \emptyset$ $(m,n \in \mathbb{N})$.

PROOF. For each $i \in \{1,\dots,m\}$ let $s_i: S^m \times S^n \to \mathbb{R}$ be the continuous map defined by $s_i(p,q) := \max\{0, e_iAq^t - pAq^t\}$. For each $j \in \{1,\dots,n\}$ let $t_j(p,q) := \max\{0, pBe_j^t - pBq^t\}$ if $(p,q) \in S^m \times S^n$. With the aid of the continuous maps $s = (s_1,s_2,\dots,s_m): S^m \times S^n \to \mathbb{R}^m$, $t = (t_1,t_2,\dots,t_n) = S^m \times S^n \to \mathbb{R}^n$ we define the map $f: S^m \times S^n \to \mathbb{R}^m \times \mathbb{R}^n$ by

$$f(p,q) = \left(\frac{p+s(p,q)}{1+\sum_{i=1}^m s_i(p,q)}, \frac{q+t(p,q)}{1+\sum_{j=1}^n t_j(p,q)}\right).$$

Now f is a continuous map from the compact convex set $S^m \times S^n$ into itself and this map possesses in view of the fixed point theorem of L.E.J. Brouwer at least one fixed point. If $(\hat{p},\hat{q})$ is such a fixed point (i.e. $(\hat{p},\hat{q}) = f(\hat{p},\hat{q})$), then it is easy to see that $s_i(\hat{p},\hat{q}) = 0$ for each $i \in \{1,\dots,m\}$ and $t_j(\hat{p},\hat{q}) = 0$ for each $j \in \{1,\dots,n\}$. From this it follows that $(\hat{p},\hat{q}) \in E(A,B) \neq \emptyset$. □

EXAMPLE 8. The mixed extension of the 4×2-matrix game in Example 7 has value $\frac{1}{3}$ and the only optimal (mixed) strategy of player I is $(0,0,\frac{2}{3},\frac{1}{3})$ (whereby with probability $\frac{2}{3}$ strategy x_3 is chosen and with probability $\frac{1}{3}$ the "bluffing strategy" x_4). The only optimal strategy for player II is $(\frac{1}{3},\frac{2}{3})$.

EXAMPLE 9. The equilibrium point set of the mixed extension of the 2×2-bimatrix game, introduced just before Proposition 1, consists of the following three points in $S^2 \times S^2$: (e_1,e_1), (e_2,e_2) and $((\frac{1}{3},\frac{2}{3}),(\frac{2}{3},\frac{1}{3}))$.

EXAMPLE 10. For the 4×6-matrix game A with payoff matrix

$$\begin{bmatrix} -1 & 1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

the value is equal to zero, $O_I(A) = \{(0,\frac{1}{2},\frac{1}{2},0)\}$ and $O_{II}(A)$ is the square in $\mathbb{R}^6$ with vertices $\frac{1}{2}(e_1+e_2)$, $\frac{1}{2}(e_1+e_6)$, $\frac{1}{2}(e_2+e_5)$ and $\frac{1}{2}(e_5+e_6)$.

The structure of the optimal strategy spaces $O_I(A)$ and $O_{II}(A)$ for finite matrix games A was clarified in the papers of SHAPLEY & SNOW [26], BOHNENBLUST, KARLIN & SHAPLEY [4] and GALE & SHERMAN [11]; all of which appeared in the first of four volumes on game theory, which were published by Princeton University Press under the titles *Contributions to the Theory of Games* I, II, III, IV. Roughly speaking, the results can be summarized as follows: the optimal strategy spaces $O_I(A)$ and $O_{II}(A)$ are polytopes, between which, there is some dimension relation and the extreme points of these polytopes correspond with certain square submatrices of A. See in connection with this also the survey paper of VOROB'EV [35]. The structure of the equilibrium point set of finite (and infinite) bimatrix games is still a subject to be studied more extensively.

Now let $A = [a_{ij}]_{i=1,j=1}^{m\ \ \infty}$ and $B = [b_{ij}]_{i=1,j=1}^{m\ \ \infty}$ be bounded $m\times\infty$-matrices ($m \in \mathbb{N}$). We call the game $<S^m,S,E_A,E_B>$, in which

$$S := \{q = (q_1,q_2,\ldots) \in \mathbb{R}^{\mathbb{N}};\ q_j \geq 0 \text{ for each } j \in \mathbb{N},\ \sum_{j=1}^{\infty} q_j = 1\},$$

$$E_A(p,q) := \sum_{i=1}^{m} \sum_{j=1}^{\infty} p_i a_{ij} q_j = pAq^t, \text{ and}$$

$$E_B(p,q) := pBq^t \qquad \text{for all } (p,q) \in S^m \times S,$$

the (*full*) *mixed extension of the semi-infinite bimatrix game* (A,B). The following theorem was proved for the first time in TIJS [31, p.95] in a different manner.

<u>THEOREM 6</u>. $E^{\varepsilon}(S^m,S,E_A,E_B) \neq \emptyset$ *for each* $\varepsilon > 0$.

<u>PROOF</u>. For each $j \in \mathbb{N}$, let $k_j := Be_j^t$ (the j-th column of B). Then $V := \{k_j;\ j \in \mathbb{N}\}$ is a bounded subset of $(\mathbb{R}^m)^t$. Let $\varepsilon > 0$. We take a finite subset W of V, such that for each $v \in V$ there is a $w \in W$ with $\|v-w\|_\infty < \varepsilon$. Without loss of generality we suppose that $W = \{k_1,\ldots,k_n\}$. (renumber the pure strategies of player II if necessary.) Take $(\hat{p},\hat{q}) \in E(A_n,B_n)$, where $A_n := [a_{ij}]_{i=1,j=1}^{m\ \ n}$, $B_n := [b_{ij}]_{i=1,j=1}^{m\ \ n}$. (That $E(A_n,B_n) \neq \emptyset$ follows from Theorem 5.)
Now let $a_n(\hat{q}) := (\hat{q}_1,\ldots,\hat{q}_n,0,0,\ldots) \in S$. We will prove that $(\hat{p},a_n(\hat{q})) \in E^{\varepsilon}(A,B)$. Firstly, we note that

(1) $$p A a_n(\hat{q})^t = pA_n\hat{q}^t \leq \hat{p}A_n\hat{q}^t = \hat{p}Aa_n(\hat{q})^t \quad \text{for each } p \in S^m.$$

Subsequently, we choose for each $r \in \mathbb{N}$ a $b(r) \in \{1,\ldots,n\}$, such that $\|k_r - k_{b(r)}\| < \varepsilon$. For each $j \in \{1,\ldots,n\}$ let $R(j) := \{r \in \mathbb{N};\ b(r) = j\}$. For each $q \in S$, let $\tilde{q} = (\tilde{q}_1,\tilde{q}_2,\ldots,\tilde{q}_n) \in S^n$ be the vector with $\tilde{q}_j = \sum_{r\in R(j)} q_r$. Then

$$Bq^t = \sum_{r=1}^{\infty} q_r k_r \leq \sum_{j=1}^{n} \Big(\sum_{r\in R(j)} q_r\Big) k_j + \varepsilon\, 1_m^t = B_n\tilde{q}^t + \varepsilon\, 1_m^t,$$

where $1_m = (1,1,\ldots,1) \in \mathbb{R}^m$. So

(2) $$\hat{p}Bq^t \leq \hat{p}B_n\tilde{q}^t + \varepsilon \leq \hat{p}B_n\hat{q}^t + \varepsilon = \hat{p}Ba_n(\hat{q})^t + \varepsilon \quad \text{for each } q \in S.$$

It follows from (1) and (2) that $(\hat{p}, a_n(\hat{q})) \in E^{\varepsilon}(A,B)$. □

In view of Proposition 1 one can immediately derive from Theorem 6 the following classic result.

THEOREM 7 (*Minimax theorem of A. Wald* [38]). *The mixed extension of each bounded semi-infinite matrix game has a value.*

REMARKS.

1. Let A be the bounded $\infty\times\infty$-matrix game of Example 4. One can easily see that the mixed extension $\langle S,S,E_A\rangle$ of this game has no value.
2. In TIJS [31] various types of mixed extensions of semi-infinite and infinite (bi-)matrix games (A,B) were studied, where also unbounded matrices A and B were allowed. For semi-infinite matrix games it was shown that all introduced mixed extensions have the same value and also the structure of the optimal strategy sets of both players was clarified. For mixed extensions of $\infty\times\infty$-matrix games different sufficient conditions were given for the existence of a value. Furthermore, it was proved that the c-mixed extension of an $m\times\infty$-bimatrix game (A,B), in which B is upper bounded, possesses ε-equilibrium points for each $\varepsilon > 0$. This last result follows also from Theorem 8 below.
3. Mixed extensions of zero-sum games on the square were studied for the first time by J. VILLE [34]. He proved that such mixed extensions have a value and optimal strategies for both players, if the

payoff function is continuous. Much information about zero-sum games on the square can be found in KARLIN [13]. However, much research can still be done on the theory of non-zero-sum games on the square. Nor has the theory of zero-sum games on the square been exhausted.

Let $\langle X,Y,K_1,K_2\rangle$ be a two-person game. For each $x \in X$ ($y \in Y$) let us denote the probability measure on X (Y) with mass 1 in x (y) by e_x (e_y). Let $P^c(X)$ be the set of all convex combinations of elements of $\{e_x;\ x \in X\}$ and similarly, let $P^c(Y)$ be the convex hull of the set $\{e_y;\ y \in Y\}$. We call the two-person game $\langle P^c(X),P^c(Y),E_1,E_2\rangle$, where

$$E_1(\mu,\nu) := \iint K_1(x,y)d\mu(x)d\nu(y)$$

$$E_2(\mu,\nu) := \iint K_2(x,y)d\mu(x)d\nu(y) \quad \text{for each } (\mu,\nu) \in P^c(X)\times P^c(Y),$$

the *c-mixed extension* of the game $\langle X,Y,K_1,K_2\rangle$. In TIJS [32] a chain of ε-equilibrium point theorems was proved, beginning with the following ε-equilibrium point theorem

<u>THEOREM 8</u>. *Let $\langle X,Y,K_1,K_2\rangle$ be a two-person game, in which X is a finite set and $K_2\colon X \times Y \to \mathbb{R}$ is an upper bounded function. Then*

$$E^{\varepsilon}(P^c(X),P^c(Y),E_1,E_2) \neq \emptyset \quad \textit{for each } \varepsilon > 0.$$

6. LINEAR PROGRAMS AND MATRIX GAMES

Many methods are developed to solve finite matrix games. With solving we mean the determination (or approximation) of the value and optimal strategies of such a game. The most successful method was suggested by John von Neumann in a conversation with G.B. Dantzig in the autumn of 1947 (cf. [7, p.277]) namely to translate the matrix game problem into a dual pair of linear programs and to solve these programs with the simplex method of G.B. Dantzig. In the same conversation von Neumann conjectured that there would be a close connection between linear programming theory and matrix game theory. This was affirmed in the papers of GALE, KUHN & TUCKER [10] and DANTZIG [6]. In this section we illuminate some aspects of the interaction between linear programs and matrix games. For books, in which attention is paid to the connection between game theory and mathematical programming, we refer to DANTZIG [7], GALE [9], KARLIN [12], OWEN [23] and STOER & WITZGALL [29].

Let A be an $m \times n$-matrix, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Let

$$P(A,b) := \{y \in \mathbb{R}^n;\ y \geq 0,\ Ay^t \leq b^t\},$$

$$D(A,c) := \{x \in \mathbb{R}^m;\ x \geq 0,\ xA \geq c\}.$$

We look at the following two (*linear programming*) *problems*:

MAIN PROBLEM. Determine $v_p(A,b,c) := \sup_{y \in P(A,b)} cy^t$ and find (if possible) an element of

$$O_p(A,b,c) := \{y \in P(A,b);\ cy^t = v_p(A,b,c)\}.$$

DUAL PROGRAM. Determine $v_d(A,b,c) := \inf_{x \in D(A,c)} xb^t$ and find (if possible) an element of

$$O_d(A,b,c) := \{x \in D(A,c);\ xb^t = v_d(A,b,c)\}.$$

$P(A,b)$, $v_p(A,b,c)$ and $O_p(A,b,c)$ are called respectively: the *feasible region*, the *value* and the *solution set of the main program* corresponding with (A,b,c). Similarly, $D(A,c)$, $v_d(A,b,c)$ and $O_d(A,b,c)$ are called the *feasible region*, the *value* and the *solution set of the dual program* corresponding with (A,b,c).

The following theorem is known as the *duality theorem for finite linear programs*.

THEOREM 9. *For* (A,b,c) *as above, precisely one of the following four assertions holds:*

1. $v_p(A,b,c) \in \mathbb{R}$, $v_d(A,b,c) \in \mathbb{R}$, $O_p(A,b,c) \neq \emptyset$, $O_d(A,b,c) \neq \emptyset$ *and there is no duality gap, i.e.* $v_p(A,b,c) = v_d(A,b,c)$.
2. *The main program is unbounded (i.e.* $v_p(A,b,c) = \infty$*) and the dual program is infeasible (i.e.* $D(A,c) = \emptyset$*).*
3. *The dual program is unbounded (i.e.* $v_d(A,b,c) = -\infty$*) and the main program is infeasible (i.e.* $P(A,b) = \emptyset$*).*
4. *Both programs are infeasible.*

The problem of solving matrix games can be translated in different manners into a pair of linear programming problems. We describe two of them in the Theorems 10 and 11.

THEOREM 10. *Let* A *be an* m×n-*matrix. Let* A^* *be the* (m+2) × (n+2)-*matrix*

$$\begin{bmatrix} A & -1_m^t & 1_m^t \\ -1_n & 0 & 0 \\ 1_n & 0 & 0 \end{bmatrix},$$

$b := (0,0,\ldots,0,-1,1) \in \mathbb{R}^{m+2}$ *and* $c := (0,0,\ldots,0,-1,1) \in \mathbb{R}^{n+2}$ $[1_m := (1,1,\ldots,1) \in \mathbb{R}^m,\ 1_n := (1,1,\ldots,1) \in \mathbb{R}^n.]$. *Then we have*

(1) $v_p(A^*,b,c) = v_d(A^*,b,c) = -v(A)$.

(2) *If* $(x,\alpha,\beta) \in O_d(A^*,b,c)$, *then* $x \in O_I(A)$.

If $(y,\gamma,\delta) \in O_p(A^*,b,c)$, *then* $y \in O_{II}(A)$.

(3) *If* $x \in O_I(A)$, *then there exist* $\alpha,\beta \in [0,\infty)$ *such that* $(x,\alpha,\beta) \in O_d(A^*,b,c)$.

If $y \in O_{II}(A)$, *then there exist* $\gamma,\delta \in [0,\infty)$ *such that* $(y,\gamma,\delta) \in O_p(A^*,b,c)$.

THEOREM 11. *Let* A *be an* m×n-*matrix. Suppose that* $v(A) > 0$. *Then we have*

(1) $v_p(A,1_m,1_n) = v_d(A,1_m,1_n) = v(A)^{-1}$.

(2) $O_d(A,1_m,1_n) = v(A)^{-1}\ O_I(A) := \{v(A)^{-1}p;\ p \in O_I(A)\}$.

$O_p(A,1_m,1_n) = v(A)^{-1}\ O_{II}(A)$.

We note that the minimax theorem of J. von Neumann can also easily be derived from the Duality Theorem 9 as follows. For a matrix game A we can look to the linear programs corresponding with (A^*,b,c), where A^*, b and c are as in Theorem 10. Note that then $P(A^*,b) \neq \emptyset$ and $D(A^*,c) \neq \emptyset$. So it follows from Theorem 9 that

$$v_p(A^*,b,c) = v_d(A^*,b,c) \in \mathbb{R},\quad O_p(A^*,b,c) \neq \emptyset,\quad O_d(A^*,b,c) \neq \emptyset.$$

Take $(x,\alpha,\beta) \in O_d(A^*,b,c)$. Then it follows that $x \in S^m$ and $xAe_j^t \geq$ $\geq -v_d(A^*,b,c)$ for each $j \in \{1,2,\ldots,n\}$. But then for the lower value of the mixed extension of A we have

$$\underline{v}(S^m,S^n,E_A) \geq \min_j xAe_j^t \geq -v_d(A^*,b,c).$$

Similarly, it follows, by taking an element $(y,\gamma,\delta) \in O_p(A^*,b,c)$, that

$$\bar{v}(S^m,S^n,E_A) \leq -v_p(A^*,b,c) = -v_d(A^*,b,c).$$

Hence the mixed extension of A has a value. Furthermore, $x \in O_I(A) \neq \emptyset$, $y \in O_{II}(A) \neq \emptyset$. Thus we have proved again the minimax theorem of J. von Neumann.

Conversely, the duality theorem can also be proved by using the following theorem in game theory. (For a proof see e.g. OWEN [23, Ch.III].)

THEOREM 12. Let A be an $m\times n$-matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. *Let* S *be the (symmetric)* $(m+n+1) \times (m+n+1)$-*matrix game with payoff matrix*

$$S = \begin{bmatrix} 0 & A & -b^t \\ -A^t & 0 & c^t \\ b & -c & 0 \end{bmatrix}.$$

Then we have

(1) *If there is a* $p \in O_I(S)$ *with* $p_{m+n+1} > 0$, *then* $P(A,b) \neq \emptyset$, $D(A,c) \neq \emptyset$ *and* $v_p(A,b,c) = v_d(A,b,c)$.

(2) *If for each* $p \in O_I(S)$ *the* $(m+n+1)$-*th coordinate equals zero, then at least one of the programs corresponding with* A, b *and* c *is infeasible.*

(3) *If* $(x_1,\ldots,x_m,y_1,\ldots,y_n,t) \in O_I(S)$ *and* $t \neq 0$, *then* $t^{-1}(x_1,\ldots,x_m) \in O_d(A,b,c)$, $t^{-1}(y_1,\ldots,y_n) \in O_p(A,b,c)$.

Now we look at the subclass $P_{m\times n}$ of linear programming problems, corresponding with triples (A,b,c), where A is an $m\times n$-matrix, where $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are vectors with all coordinates positive and where $D(A,c) \neq \emptyset$. For $(A,b,c) \in P_{m\times n}$ also $P(A,b) \neq \emptyset$ because $0 \in P(A,b)$. Hence it follows from Theorem 9 that

$$v_p(A,b,c) = v_d(A,b,c) \in \mathbb{R},\ O_p(A,b,c) \neq \emptyset,\ O_d(A,b,c) \neq \emptyset.$$

Now we provide $P_{m\times n}$ with the metric d, defined by

$$d((A,b,c),(A',b',c')) := \max\{\|A-A'\|_\infty, \|b-b\|_\infty, \|c-c'\|_\infty\}$$

and ask ourselves what is the influence of perturbations on value and solution sets. The following holds:

THEOREM 13. *The value function* $v_d\colon P_{m\times n} \to \mathbb{R}$ *is a continuous function.* $O_p\colon P_{m\times n} \to \mathbb{R}^n$, $O_d\colon P_{m\times n} \to \mathbb{R}^m$ *are upper semicontinuous multifunctions.*

With some labour one can give a direct proof of this theorem (see [33, p.207] where a generalization of this theorem to semi-infinite programs is proved). A more simple and less technical proof can be obtained by using the following well known theorem in game theory, with a simple proof (cf. [12]).

THEOREM 14. *Let* $M_{m\times n}$ *be the set of* $m\times n$-*matrices, provided with the* $\|\cdot\|_\infty$-*norm. Then*

$$\begin{aligned}
v \;&: M_{m\times n} \to \mathbb{R} \text{ is a continuous function}\\
O_I \;&: M_{m\times n} \to S^m \text{ is an upper semicontinuous multifunction}\\
O_{II}&: M_{m\times n} \to S^n \text{ is upper semicontinuous.}
\end{aligned}$$

Now Theorem 13 follows from Theorem 14 (and conversely) by assigning to triples $(A,b,c) \in P_{m\times n}$ the $m\times n$-matrices $\hat{A} := [a_{ij}/b_ic_j]_{i=1}^{m}{}_{,j=1}^{n}$, and by observing that the following hold:

(1) $v_d(A,b,c) = v_d(\hat{A},1_m,1_n) = v(\hat{A})^{-1}$.

(2) $O_I(\hat{A}) = \{v(\hat{A})x;\ x \in O_d(\hat{A},1_m,1_n)\}$,
$O_{II}(\hat{A}) = \{v(\hat{A})y;\ y \in O_p(\hat{A},1_m,1_n)\}$.

(3) $O_p(\hat{A},1_m,1_n) = \{(c_1y_1,c_2y_2,\ldots,c_ny_n);(y_1,\ldots,y_n) \in O_p(A,b,c)\}$,
$O_d(\hat{A},1_m,1_n) = \{(b_1x_1,b_2x_2,\ldots,b_mx_m);(x_1,\ldots,x_m) \in O_d(A,b,c)\}$.

The connection between semi-infinite linear programming problems and semi-infinite matrix games was studied by SOYSTER [28] and TIJS [33]. With an $m\times\infty$-matrix A, a $b \in \mathbb{R}^m$ and a $c = (c_1,c_2,\ldots) \in \mathbb{R}^{\mathbb{N}}$, again one can associate dual pairs of linear programs. To avoid convergence problems we take here for $P(A,b)$ the set $\{y \in \mathbb{R}_c^{\mathbb{N}};\ y \geq 0,\ Ay^t \leq b^t\}$, where $\mathbb{R}_c^{\mathbb{N}}$ consists of the infinite sequences of real numbers, where only a finite number of coordinates may be unequal to zero. However, extending the duality theory for finite programs to infinite programs cannot be straightforward as can be seen from the following example (cf. [33, p.199]).

EXAMPLE 11. Let $b := (1,0)$, $c := (1,2,2,\dots)$ and

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & \dots \end{bmatrix}.$$

Then

$$P(A,b) = \{y \in \mathbb{R}^{\mathbb{N}};\ 0 \le y_1 \le 1,\ y_j = 0 \text{ for each } j > 1\},$$
$$D(A,c) = \{x \in \mathbb{R}^2;\ x_1 \ge 2,\ x_2 \ge 0\}.$$

Hence there is a duality gap:

$$v_p(A,b,c) = 1 \ne 2 = v_d(A,b,c).$$

One can derive the following duality theorem (cf. TIJS [33,p.203]).

THEOREM 15. *Let* (A,b,c) *be as above. Suppose further that all coordinates of* b *are positive. Then exactly one of the following two assertions holds:*

(1) $D(A,c) = \emptyset$ *and* $v_p(A,b,c) = \infty$ $(= v_d(A,b,c))$.

(2) $v_p(A,b,c) = v_d(A,b,c) \in [0,\infty)$ *and* $O_d(A,b,c) \ne \emptyset$.

With the aid of this theorem one can prove that the c-mixed extension of a semi-infinite matrix games has a value (cf. [33, p.17]).

We conclude this section with one of the many results which connect non-linear programming problems with the theory of zero-sum games, namely:

THEOREM 16 (*Theorem of KARLIN,* [12, p.201]). *Let* C *be a convex subset of* $\mathbb{R}^m$ *and let* $g: C \to \mathbb{R}$, $f_j: C \to \mathbb{R}$ $(j \in \{1,2,\dots,n\})$ *be* $n+1$ *concave functions on* C*, such that*

$$\forall_{y \in \mathbb{R}^n_+ - \{0\}}\ \exists_{x \in C} \left[\sum_{j=1}^{n} y_j f_j(x) > 0 \right].$$

Let

$$D := \{x \in C;\ f_j(x) \ge 0 \text{ for each } j \in \{1,\dots,n\}\}.$$

Look on the one side at the non-linear programming problem:

$$\text{NLP: find } x^* \in D, \text{ such that } g(x^*) = \sup_{x \in D} g(x),$$

and on the other hand, at the two-person zero-sum game

$$\langle D, \mathbb{R}^n_+, K\rangle,$$

where

$$K(x,y) := g(x) + \sum_{j=1}^{n} y_j f_j(x).$$

Then we have the following connection:

(1) If x^* is a solution of NLP, then there exists a $y^* \in \mathbb{R}^n_+$, such that (x^*,y^*) is a saddlepoint of the game $\langle D, \mathbb{R}^n_+, K\rangle$.

(2) If (x^*,y^*) is a saddlepoint of the above mentioned game, then x^* is a solution of problem NLP.

7. LINEAR COMPLEMENTARY PROBLEMS AND BIMATRIX GAMES

It is not surprising that the theory of complementarity problems has drawn much attention the last fifteen years in Operations Research, Mathematical Economy and Game Theory. Many standard problems in those areas can be translated into a (linear or non-linear) complementarity problem. In this short section we only want to indicate the relation between the problem of finding equilibrium points of bimatrix games and a linear complementarity problem. For a thorough introduction into complementarity theory and for applications we refer to BASTIAN [1] and LÜTHI [18].

For a given $f: \mathbb{R}^k \to \mathbb{R}^k$ we mean by the *complementarity problem corresponding with* f, the following problem.

CP: Find $x \in \mathbb{R}^k$, such that $x \geq 0$, $f(x) \geq 0$ and $x(f(x))^t = 0$.

If f is an affine function, then the problem is called a *linear complementarity problem*.

In the following theorem a one to one correspondence is given between the equilibrium points of a finite bimatrix game and the solutions of a certain linear complementarity problem.

THEOREM 17. *Let* A *and* B *be* $m\times n$*-matrices and suppose that* $A > 0$ *and* $B < 0$. *Let* $f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ *be defined by*

$$(f(x,y))^t = \begin{bmatrix} 1_m^t \\ -1_n^t \end{bmatrix} + \begin{bmatrix} 0 & -A \\ -B^t & 0 \end{bmatrix} \begin{bmatrix} x^t \\ y^t \end{bmatrix}$$

and let

$$\underline{O} := \{(x,y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n;\ f(x,y) \geq 0,\ (x,y)(f(x,y))^t = 0\}.$$

Then we have

(1) *If* $(p,q) \in E(A,B)$, *then* $(-(pBq^t)^{-1}p,\ (pAq^t)^{-1}q) \in \underline{O}$.

(2) *If* $(x,y) \in \underline{O}$, *then* $((\sum_{i=1}^m x_i)^{-1}x,\ (\sum_{j=1}^n y_j)^{-1}y) \in E(A,B)$.

A well known algorithm for the solution of linear complementarity problems is the algorithm of LEMKE (cf. [18] and [16]) and this can thus in view of Theorem 17 be used to find equilibrium points of bimatrix games. As a matter of fact, the bimatrix game problem was the starting point of the theory of complementarity problems.

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